

difference, divided by Carnot's function, which is $\frac{J}{t}$, if the temperature is measured on our absolute scale. Hence if a film such as a soap-bubble be enlarged, its area being augmented in the ratio of 1 to m , it experiences a cooling effect, to an amount calculable by finding the lowering of temperature produced by removing a quantity of heat equal to

$$m \frac{J}{J} \times \frac{-dT}{dt},$$

from an equal mass of liquid unchanged in form.

For water $T=2.96$ gr. per lineal inch.

Work per square inch spent in drawing out a film $=5.92$, say 6 grains, $\frac{dT}{dt} = \frac{1}{550}T$, or thereabouts.

Suppose $\frac{t'}{J} = \frac{300}{1390 \times 12}$, then the quantity of heat to be removed, to produce the cooling effect, per square inch of surface of augmentation of film will be $\frac{1}{5100}$. Suppose, then, 1 grain of water to be drawn out to a film of 16 square inches, the cooling effect will be $\frac{16}{5100}$ of a degree Centigrade, or about $\frac{1}{320}$. The work spent in drawing it out is $16 \times 6 = 96$ grains and is equivalent to a heating effect of $\frac{96}{12 \times 1390} = \frac{1}{174}$. Hence the total energy (reckoned in heat) of the matter is increased $\frac{1}{174} + \frac{1}{320}$ of a degree Centigrade, when it is drawn out to 16 square inches.

IV. "On the Logocyclic curve, and the geometrical origin of Logarithms." By the Rev. J. BOOTH, LL.D., F.R.S.
Received April 15, 1858.

In a paper read before the Mathematical Section of the British Association during its meeting at Cheltenham in 1856, and which was printed among the reports for that year, I developed at some length the geometrical origin of logarithms, and showed that a trigonometry exists as well for the parabola as for the circle, and that every formula in the latter may be translated into another which shall indicate some property of parabolic arcs analogous to that from which it has been derived. I showed, moreover, that the

whole theory of logarithms was founded on a basis as purely geometrical as the trigonometry of the circle, and I pointed out how the imaginary expressions in the latter—such as De Moivre's theorem—have real counterparts in the trigonometry of the parabola. As the principle of duality is of the widest application in geometrical investigation, and as every property of circumscribed space has its dual, it would be strange if the dual of circular trigonometry had no existence.

In the paper to which I have referred, I showed how, by the help of certain arbitrary lines drawn about the parabola, numbers and their logarithms might be exhibited. But as there was something conventional in this representation, I was not quite satisfied with the construction. I suspected that the geometrical theory of logarithms was just as little conventional as the trigonometry of the circle. With this view I have again lately considered the whole subject, and by the help of a new curve, which I have called the *Logocyclic Curve*, from the similarity of many of its properties to those of the circle, and from its use in representing numbers and *their logarithms*, I have succeeded in exhibiting the whole theory of logarithms in a geometrical form as complete as it is comprehensive, and simple as it is beautiful.

The properties of the Logocyclic curve, a curve which hitherto has escaped discussion, if not discovery, are many of them as remarkable as they are simple, and I will now proceed to mention some of the most obvious.

The equations of the Logocyclic curve.

Let r and θ be the polar coordinates of the curve, then its equation is

$$r = a(\sec \theta \pm \tan \theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since $\sec \theta = \frac{r}{x}, \tan \theta = \frac{y}{x}, r = \sqrt{x^2 + y^2},$

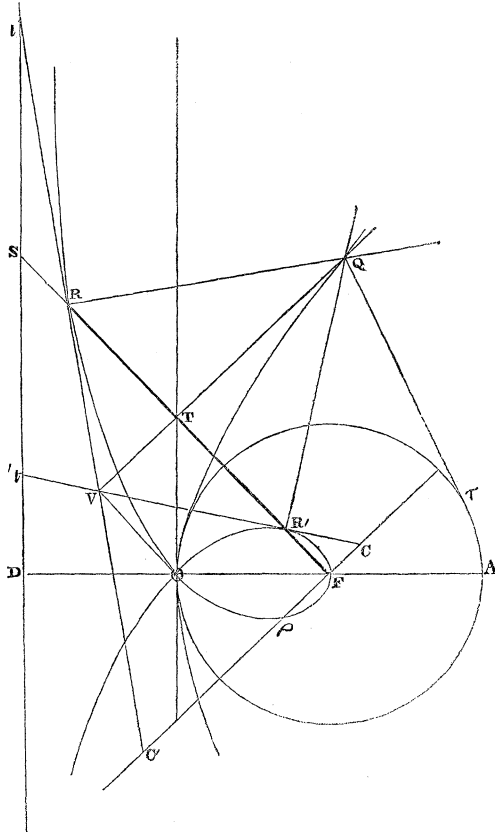
we get for the equation of the curve in rectangular coordinates,

$$y^2(2a - x) = x(a - x)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or $(x^2 + y^2)(2a - x) = a^2 x.$

Let the focus F of the parabola, whose vertical focal distance FO is a , be taken as the origin, the axis and parameter of the parabola being the axes of x and y ; let the vertical tangent OT and the directrix of the parabola DS be drawn, then the following properties of this curve may easily be shown :—

I. The directrix DS of the parabola is an asymptote of the curve, which has two infinite branches along this line above and below the axis. These branches intersect at right angles on the axis of the parabola in its vertex, and form an oval or loop between this point and the focus of the parabola F .



II. The radius vector $r = a(\sec \theta \pm \tan \theta)$, drawn from the focus F of the parabola, cuts the loop and one of the infinite branches in the points R_1 and R , so that

$$FR = a(\sec \theta + \tan \theta), \text{ and } FR_1 = a(\sec \theta - \tan \theta).$$

The product of the segments of this secant, as in the circle, is constant and equal to a^2 .

These points may be named *Reciprocal Points*.

III. O being the vertex of the parabola, in which the two branches of the curve cut at right angles, the angle ROR' is always a right angle.

IV. Through the points R and R_1 let normals be drawn to the Logocyclic curve, they will meet the parabola in the same point Q , and the parabolic arc OQ diminished by the protangent QT is the logarithm of the number FR or of its reciprocal FR_1 .

Q , the intersection on the parabola of the normals, drawn to the Logocyclic curve at the *reciprocal* points R , R_1 may be called the Logarithmic point of the line FR .

V. The normals RQ and R_1Q are equal. This is also a property of the circle.

VI. At the points R and R_1 , the *reciprocal points*, let tangents be drawn to the curve and meet in V . The tangents RV and R_1V are equal, and they are equally inclined to the chord RR_1 . This is another property of the circle.

VII. The locus of V , the intersection of tangents drawn to the curve through the *reciprocal points*, is the Cissoïd of Diocles, a curve discovered in the Schools of Alexandria.

VIII. Since the line VQ is at right angles to the line VO , we have this new property of that old curve, that a line drawn through any point of a Cissoïd at right angles to the radius vector drawn to the cusp of the curve envelopes a parabola.

IX. The directrix of the parabola is the asymptote of the Cissoïd, and its cusp is at O the vertex of the parabola.

X. The sum of the ordinates of the reciprocal points R and R_1 is equal to the ordinate of the logarithmic point Q on the parabola, and the sum of the distances of the points R and R_1 from the asymptote DS is constant and equal to $2a$.

XI. The sum of the products of the abscissæ and ordinates of the *reciprocal points* on the curve is constant, or

$$x_1x_{11} + y_1y_{11} = a^2.$$

XII. The distances of any point Q on a parabola from its focus and directrix are equal. We may generalize this well-known theorem, and say the distances are equal of any point on a parabola from its Logocyclic curve, measured along the two normals to this curve drawn through the point Q .

XIII. As the tangent to a parabola makes equal angles with the focal radius vector and a perpendicular on the directrix, so also this tangent makes equal angles with the normals to the Logocyclic curve drawn through this point of contact.

XIV. The two reciprocal points R and R_1 on the Logocyclic, the logarithmic point Q on the parabola, and the intersection V of the tangents to the Logocyclic which meet on the Cissoid lie on the circumference of a circle, or in other words, any two reciprocal points on the curve, the intersection of the normals at these points and the intersection of the tangents at the same points lie in the circumference of a circle.

XV. Through the two *reciprocal points* R and R_1 a circle may be drawn touching the axis at O , and having its centre at T the intersection of the vertical tangent OT , with the radius vector FR . This property leads to a simple construction of the curve by points.

XVI. The tangent to the curve at either of the *reciprocal points* makes with the radius vector through this point an angle whose tangent is equal to the cosine of the inclination of the radius vector to the axis.

XVII. The sum of the polar subtangents FC and FC_1 belonging to the reciprocal points R and R_1 is constant, and equal to $2a$.

XVIII. The sum of the reciprocals of the polar subnormals belonging to two reciprocal points is constant, and equal to $\frac{2}{a}$.

XIX. The lengths of the tangents to the curve between any two reciprocal points and the asymptote are equal, or $Rt = R_1t_1$, and $St = St_1 = a \sec \theta$.

XX. The products of these two tangents t , and the perpendiculars p, p_1 from the focus F upon them, is constant, or $t^2 p p_1 = a^4$.

XXI. Let ρ and ρ_1 be the reciprocals of the radii of curvature of the Logocyclic curve at the points R and R_1 ; and let ρ_{11} be the reciprocal of the radius of curvature of the parabola at a point through which the normal makes the angle ψ with the axis, then

$$\rho + \rho_1 = 8\rho_{11}$$

ψ and θ are connected by the condition $\tan \psi = \cos \theta$. See (XV.)

XXII. The Logocyclic curve is the envelope of all the circles whose centres range along the parabola, and whose radii are successively equal to $\sqrt{f^2 - a^2}$, f being the distance of the centre, Q , of the

circle from the focus of the parabola. This follows from an inspection of the figure; but it may easily be proved as an independent theorem.

XXIII. The vertical tangent OT bisects all the cords of the Logocyclic passing through F. The angles VRT or VR₁T = ψ and θ or OFR are so connected that $\tan \psi = \cos \theta$. Hence the maximum ordinate of the loop is found by making $\psi = \theta$, or $\tan \theta = \cos \theta$, or $\sin \theta = \frac{\sqrt{5}-1}{2}$.

XXIV. If any point Q on the parabola be taken as centre, and through the two corresponding reciprocal points on the Logocyclic curve a circle be drawn, it will always cut at right angles the fixed circle whose centre is the focus F and radius = a .

XXV. The Logocyclic curve, like the circle, is its own *inverse* curve.

On the Area and rectification of the Logocyclic Curve.

XXVI. The area of the loop will be found equal to

$$2a^2 - \frac{a^2\pi}{2},$$

and the area between the asymptote and infinite branch will be

$$2a^2 + \frac{a^2\pi}{2}.$$

Hence the entire area of the curve is $4a^2$, or equal to the square of the semiparameter of the parabola. It is obvious that the area of the half-loop is equal to the difference between the square of a and the quadrant of a circle inscribed in it, while that of the infinite branch is equal to the square of a with the quadrant added to it. Hence also the difference of the two areas is equal to $a^2\pi$, that is to the area of a circle whose radius is a .

The length of the half-loop, together with that of the infinite branch of the curve, is equal to the infinite branch of an equilateral hyperbola whose transverse semiaxis is $\sqrt{2}a$, and to the half-loop of the lemniscate which belongs to that equilateral hyperbola. Hence the entire length of the Logocyclic is equal to the entire continuous arc of an equilateral hyperbola whose transverse axis is $\sqrt{2}a$, and to the loop of the lemniscate which belongs to that branch; while the difference between the lengths of the loop and the infinite branch is equal to an arc of the parabola together with a right line.

To represent Numbers and their Logarithms by the Logocyclic Curve and its Conjugate Parabola.

XXVII. A parabola whose focal vertical distance is a or 1 being drawn, and also its Logocyclic curve, let a radius vector be drawn to the latter equal to the given number n . Then $n = \sec \theta + \tan \theta$.

Let this line meet the vertical tangent in T, the parabolic arc OQ—QT is the logarithm of n .

It is clear that the infinite branch of the curve from $+\infty$ to O will give radii vectores of every magnitude from ∞ to 1, and parabolic arcs from ∞ to 0; hence, while the numbers range from ∞ to 1, the parabolic arcs range from ∞ to 0. When the number lies between 1 and 0, the radius vector representing it is drawn *below* the axis; its extremity will be found on the loop, and the corresponding arc of the parabola will be negative, hence the logarithm of a positive number is equal to the logarithm of its reciprocal, with the sign changed; for the magnitude of the parabolic arc depends on θ , and θ is the same in $\sec \theta + \tan \theta$, as in its reciprocal $\sec \theta - \tan \theta$.

Hence while the infinite branch of the Logocyclic curve from $+\infty$ through R, O, ρ , to F, may by its radii vectores represent all positive numbers from $+\infty$ to $+0$, the *two* infinite branches of the parabola will be used in representing the logarithms of positive numbers from $+\infty$ to $+0$; that is, the upper or positive branch of the parabola will be “used up” in representing the logarithms of positive numbers from $+\infty$ to $+1$, and the lower or negative branch of the parabola in representing the logarithms of positive fractional numbers from $+1$ to $+0$. Hence there is no construction by which we can represent negative numbers or their logarithms, therefore such numbers can have no logarithms.

Let radii vectores be drawn from F to the Logocyclic curve equal to $e, e^2, e^3, e^4 \dots e^n$, then these lines will meet the tangent to the vertex of the parabola in the points T, $T_\rho, T_{\rho\rho} \dots T_n$; and tangents being drawn from these points, touching the parabola in Q, $Q_\rho, Q_{\rho\rho}, Q_{\rho\rho\rho}, Q_n$, the logarithms of these numbers will be

$$OQ - QT = 1, \quad OQ_\rho - Q_\rho T_\rho = 2, \quad OQ_{\rho\rho} - Q_{\rho\rho} T_{\rho\rho} = 3, \dots$$

$$OQ_n - Q_n T_n = (n+1);$$

hence the logarithms of e, e^2, e^3, e^n are 1, 2, 3, $\dots n$.

In like manner we should find the logs of $e, e^{\frac{1}{2}}, e^{\frac{1}{3}}, e^{\frac{1}{4}} \dots e^{\frac{1}{n}}$ to be

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots \frac{1}{n}.$$

Let a series of radii vectores be drawn from the point F to the Logocyclic curve in geometrical progression, and let them be

$$(\sec \theta + \tan \theta), (\sec \theta + \tan \theta)^2, (\sec \theta + \tan \theta)^3, \dots (\sec \theta + \tan \theta)^n,$$

meeting the vertical tangent to the parabola in the points $T_1, T_2, T_3, \dots T_n$, and let the tangents drawn from the points $T_1, T_2, \dots$ touch the parabola in the points $Q_1, Q_2, Q_3, \dots Q_n$; let the difference between the first parabolic arc and its *protangent* be δ , then we shall have $OQ_1 - Q_1T_1 = \delta$, $OQ_2 - Q_2T_2 = 2\delta$, $OQ_3 - Q_3T_3 = 3\delta$ $OQ_n - Q_nT_n = n\delta$.

Or while numbers increase in geometrical progression, their logarithms increase in arithmetical progression.

As every number whose logarithm is to be exhibited must be put under the form $\sec \theta + \tan \theta$, which is of the form $1 + x$, since the limiting value of $\sec \theta$ is 1, we discover the reason why in developing the logarithm of a number the number itself must be put under the form $1 + x$, or some derivative from it, and not simply under that of x .

If we equate $\sec \theta + \tan \theta$ with $1 + x$, we shall find

$$x = \frac{2 \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}}. \quad \text{Let } u = \tan \frac{\theta}{2},$$

then $n = \sec \theta + \tan \theta = 1 + x = \frac{1+u}{1-u}$, which is another familiar form under which a number is put, whose logarithm is to be developed in a series.

Let θ be the angle which the line $(\sec \theta + \tan \theta)$ makes with the axis, and let $\theta_1, \theta_2, \theta_3, \dots$ be the angles which the lines $(\sec \theta + \tan \theta)^2, (\sec \theta + \tan \theta)^3, \dots (\sec \theta + \tan \theta)^n$ make with the same axes, then

$$\begin{aligned} \theta &= \theta \\ \theta_1 &= \theta + \theta \\ \theta_2 &= \theta + \theta + \theta \\ &\dots \\ \theta_n &= \theta + \theta + \theta + \theta \dots \text{to } n \text{ terms.} \end{aligned}$$

The *definitions* of the symbols $+$, $-$, which I call logarithmic or parabolic plus and minus, are given in the paper referred to.

Hence

$$(\sec \theta + \tan \theta)^n = \sec(\theta + \theta + \theta \dots n) + \tan(\theta + \theta + \theta \dots \text{to } n).$$

Change $\sec \theta$ into $\cos \phi$, $\tan \theta$ into $\sqrt{-1} \sin \phi$, and $+$ into $-$, then we shall have

$$(\cos \phi + \sqrt{-1} \sin \phi)^n = \cos n\phi + \sqrt{-1} \sin n\phi.$$

Hence De Moivre's theorem, which represents an imaginary property of the circle, has its counterpart in a real property of the parabola.

The base of the Neperian system e is that particular radius vector drawn to the Logocyclic which gives the difference between the corresponding parabolic arc and its *protangent* equal to the focal vertical distance of the parabola. Let ϵ be the angle which this radius vector makes with the axis. Then

$$e = \sec \epsilon + \tan \epsilon.$$

$$e = 2.718281828.$$

Hence also as the angles $\theta, \theta + \theta, \theta + \theta + \theta, \dots n\theta$ give circular arcs which increase in arithmetical progression, the angles $\theta, \theta + \theta, \theta + \theta + \theta, \&c.$ give parabolic arcs, whose excesses over their *protangents* increase in arithmetical progression.

If the lines $(\sec \theta + \tan \theta), (\sec \theta + \tan \theta)^2, (\sec \theta + \tan \theta)^3, \&c.$ were drawn, making *equal* angles with each other, and therefore multiples of θ with the axis, instead of the angles $\theta; \theta + \theta, \theta + \theta + \theta$, the locus of the extremities would be the logarithmic spiral instead of the Logocyclic curve.

The spiral of Archimedes may also be used as a means of exhibiting logarithms or parabolic arcs. For the equation of the spiral being $r = a\theta$, the arc of the spiral is given by the equation $s = a \int \sqrt{1 + \theta^2} d\theta$.

Let $a\theta = a \tan \phi$, then making the necessary substitutions, $s = a \int \frac{d\phi}{\cos^3 \phi}$; but this is the expression for an arc of a parabola measured from the vertex to a point whose ordinate is $2a \tan \phi = 2r$, that is to twice the radius vector of the spiral.

Hence, if a line be drawn along the vertical tangent of a parabola equal to twice the radius vector of the spiral, and a line be drawn parallel to the base, it will determine the parabolic arc corresponding to the radius vector of the spiral.

It is not difficult to construct a trammel which shall describe the

Logocyclic Curve by a continuous motion ; and a very ingenious instrument has been contrived by Mr. Henry Johnson of Crutched Friars, to describe the spiral of Archimedes, which is as simple as it is effective.

June 17, 1858.

The LORD WROTTESELEY, President, in the Chair.

The Earl Granville, Professor Hennessy, and the Rev. Samuel Haughton were admitted into the Society.

In accordance with notice given at the last Meeting, the Earl of Rosse proposed the Right Hon. Sir John Pakington, Bart. for election and immediate ballot.

The Ballot having been taken, Sir John Pakington was declared duly elected.

The following communications were read :—

I. "On the Problem of Three Bodies." By CHARLES JAMES HARGREAVE, LL.D., F.R.S. Received May 3, 1858.

(Abstract.)

The author states that the principal object of this memoir is to set forth two new methods of treating the dynamical equations by processes of variation of elements, differing from the ordinary processes of this nature principally in this particular, that the variations are represented in explicit terms of the elements themselves and of the time, and not through the medium of partial differential coefficients. It has been his object to render the processes as elementary as possible ; and to preserve them in a vigorous form, by postponing all attempts at approximation until the formulæ are actually applied to practical problems. The applications given in the paper comprise the circular and spherical pendulums, and the planetary and lunar theories, and a special theorem as to the movement of the plane of a planet's motion under the influence of several other planets.

The original normal problem which is taken as the basis, is that